Linear programming models for a stochastic dynamic capacitated lot sizing problem

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A B S T R A C T
In this paper the stochastic dynamic lot sizing problem with multiple items and limited capacity under two types of fill rate constraints is considered. It is assumed that according to the static-uncertainty strategy of Bookbinder and Tan [2], the production periods as well as the lot sizes are fixed in advance for the entire planning horizon and are executed regardless of the realisation of the demands. We propose linear programming models, where the non-linear functions of the expected backorders and the expected inventory on hand are approximated by piecewise linear functions. The resulting models are solved with a variant of the Fix-and-Optimize heuristic. The results are compared with those of the column generation heuristic proposed by Tempelmeier [14].

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1. Introduction
Lot sizing problems occur in industrial practice, when a production process can only start after a setup of the required resources with associated setup time and/or setup costs has been completed. In the literature, numerous lot sizing approaches for different production situations have been proposed. Particularly relevant for operational production planning in manufacturing companies are dynamic lot sizing models, as they consider demands and orders of varying sizes associated with specific due dates.

The majority of the lot sizing literature focusses on the situation when all data are deterministically known in advance. Industrial planning practice usually applies a forecasting procedure that provides a deterministic time series of the expected future demands. Uncertainty is taken into consideration by reserving a fixed amount of inventory as safety stock. The amount of this reserve stock is usually computed by simple rules of thumb, e.g. the standard deviation of the demand during the risk period is multiplied by a quantile of the standard normal distribution. In this way, it is usually not possible to meet a target service level. In addition, the effect of the lot sizes on the risk hedging is not taken into consideration. For example, with large lot sizes it may be optimal to provide no safety stock at all.

In this paper, we consider the stochastic dynamic multi-item capacitated lot sizing problem (SCLSP), which is the stochastic counterpart of the well-known deterministic multi-item capacitated lot sizing problem (CLSP). The problem can be described as follows. We consider a single resource, which is used to produce $K$ items with dynamic random period demands $d_{kt}$ over a planning horizon of $T$ periods. For product $k$, the demands $d_{kt}$ are random variables with forecasted period-specific expectation $E(D_{kt})$ and variance $V(D_{kt})$. The period capacities of the resource are $b_t$ ($t = 1, 2, \ldots, T$).

We assume that the “static-uncertainty strategy” according to [2] is in place, which means that at the beginning of the planning horizon the complete production plan is fixed, including the timing and the size of production quantities. Unlike the “dynamic-uncertainty strategy” and the “static-dynamic-uncertainty strategy” which result in random lot sizes, this strategy has the advantage that it is possible to construct a production plan that respects limited capacities with certainty.

This problem definition reflects the scenario that can be observed in MRP planning environments and in so-called Advanced Planning Systems. With given setup $s_k$ and holding costs $h_k$ ($k = 1, 2, \ldots, K$), we seek to determine production quantities to satisfy the time-varying random period demands and minimize the sum of setup and holding costs. Inventory holding costs are charged on the inventory at the end of each period. As backlog costs are usually difficult if not impossible to quantify, we assume that a $\beta$ service level (fill rate) is used as a performance criterion.

The $\beta$ service level relates the total amount backordered to the total demand observed during a given time span, whereby the
backorder in period \( t \), \( B_t \), depends on the demand observed in period \( t \), \( D_t \), and the available inventory at the beginning of period \( t \). In stochastic inventory systems, under stationary conditions, usually the long-term average fill rate is considered. However, the \( \beta \) service level can also be calculated for any finite number of periods (see [1,4,17]). Let \( Y^{(1)}_t \) be the cumulated demand and let \( B^{(1)}_t \) be the cumulated backorders from period 1 to \( t \). Then the fill rate calculated w.r.t. periods 1 to \( t \), \( \beta_t \), is defined as \( \beta_t = 1 - B^{(1)}_t / Y^{(1)}_t \).

As an alternative to \( \beta_t \), the cycle service level \( \beta_c \) relates the backorders within a replenishment cycle to the demand that occurs in that cycle. This criterion can only be calculated if the cycle length is known. If two production orders are released in, for example, periods 3 and 6, then the coverage time of the first order runs from period 3 to 5. For the calculation of the \( \beta_c \) service level the backorders, which newly occur in periods 3 to 5 are added and related to the demands of these three periods. A \( \beta_c \) service level constraint is more restrictive than a \( \beta \) constraint, because it requires the achievement of the target in every cycle. A bad performance in one cycle cannot be compensated by a good performance in a different cycle, which would be possible with the \( \beta \) criterion.

It can be shown that \( \beta_c \) can be expressed in terms of cumulated backorders and cumulated demands. If two consecutive order cycles \((i-1)\) and \( i \) end in periods \( \tau_{i-1} \) and \( \tau_i \), respectively, then \( \beta_c(\tau_i) \) is defined as

\[
\beta_c(\tau_i) = 1 - \frac{B^{(i)}_{\tau_i} - B^{(i-1)}_{\tau_{i-1}}}{Y^{(i)}_{\tau_i} - Y^{(i-1)}_{\tau_{i-1}}}
\]

where the numerator describes the net backorders, which newly occurred in cycle \( i \) and the denominator is the corresponding demand. There is an interesting relation between \( \beta_c \), \( \beta_{\tau_{i-1}} \), and \( \beta_c(\tau_i) \), which is useful for the formulation of a lot sizing model. At the end of period \( \tau_i \), the finite period service level is defined as

\[
\beta_{\tau_i} = 1 - \frac{B^{(i)}_{\tau_i}}{Y^{(i)}_{\tau_i}}
\]

Similarly, at the end of period \( \tau_{i-1} \), we have

\[
\beta_{\tau_{i-1}} = 1 - \frac{B^{(i-1)}_{\tau_{i-1}}}{Y^{(i-1)}_{\tau_{i-1}}}
\]

Now, assume that the production quantities are set such that at the end of each production cycle the finite period service levels are equal: \( \beta_{\tau_i} = \beta_{\tau_{i-1}} = \beta \).

Then

\[
(1-X) \cdot Y^{(i)}_{\tau_i} = B^{(i)}_{\tau_i}
\]

and

\[
(1-X) \cdot Y^{(i-1)}_{\tau_{i-1}} = B^{(i-1)}_{\tau_{i-1}}
\]

Taking the difference between [4] and [5], we obtain

\[
(1-X) \cdot \left( Y^{(i)}_{\tau_i} - Y^{(i-1)}_{\tau_{i-1}} \right) = \left( B^{(i)}_{\tau_i} - B^{(i-1)}_{\tau_{i-1}} \right)
\]

or

\[
1-X = \frac{B^{(i)}_{\tau_i} - B^{(i-1)}_{\tau_{i-1}}}{Y^{(i)}_{\tau_i} - Y^{(i-1)}_{\tau_{i-1}}} = 1 - \beta_c(\tau_i).
\]

Hence, if lot sizes are set such that \( \beta_c = \beta \), then \( \beta_c(\tau_i) = \beta_c = \beta \). As a consequence, it is possible to meet a \( \beta_c \) service level target through the introduction of surrogate \( \beta \) service level constraints in a lot sizing model. This is what we are proposing in this paper. Thereby, in the constraints of the model, we quantify the actual service level by the ratio of the expected values of the backorders and the demands, which is only exact for an infinite time horizon. As for a limited time horizon \( t \) the relation \((1 - E(B/Y)) \geq (1 - E(B/E(Y)))\) holds [1], this is a conservative approximation which ensures that the target set by the management will be met.

The rest of this paper is organized as follows. In Section 2 the relevant literature is reviewed. In Section 3 we describe the problem in detail. The model formulations are presented in Section 4. Following, the solution approach is presented in Section 5. The results of a numerical experiment are reported in Section 6. Finally, Section 7 contains some concluding remarks.

### 2. Literature

In the literature, we observe a rapidly increasing amount of papers on stochastic lot sizing problems. However, only a limited number of researchers have considered dynamic capacitated lot sizing problems with random demand and service level constraints. Reviews of stochastic lot sizing problems which deal with multiple items produced on a single resource with limited capacity are presented by Sox et al. and Winandts et al. [10,18] and in chapter E in [15]. Sox and Muckstadt [11] solve a variant of the stochastic dynamic CLSP, where item- and period-specific backorder costs as well as extendible production capacities are considered. The authors propose a Lagrangean heuristic to solve the resulting non-linear integer programming problem that is repeatedly applied in a dynamic planning environment. Brandimarte [3] considers the stochastic CLSP where the uncertainty of the demand is represented by a scenario tree. In this case, the period demands are modeled as discrete random variables. The evolution of demand over time is depicted with a directed layered tree, where each layer corresponds to a planning period and the nodes are linked to realizations of the discrete stochastic demand process. The resulting large-scale deterministic MIP model is then solved with a commercially available solver using rolling schedules with lot sizing windows. As demonstrated by Brandimarte [3], the scenario-based approach suffers from a dramatically increasing complexity, if the number of periods and/or the number of possible outcomes of the period demands are increased. In addition, currently there are no scenario-based models available which could account for product-specific fill rate constraints. Tempelmeier and Herpers [16] propose a formulation of the dynamic capacitated lot sizing problem under random demand, when the performance is measured in terms of a fill rate per cycle \( \beta_c \). They propose the ABC\(_c\) heuristic which is an extension of the A/B/C heuristic proposed for the solution of the deterministic CLSP by Maes and van Wassenhove [8]. A different solution approach which outperforms the ABC\(_c\) heuristic is presented in [14]. The author proposes a heuristic solution procedure that combines column generation and the ABC\(_c\) heuristic. Helber et al. [7] consider a formulation of the stochastic CLSP with a so-called \( \delta \) service level constraint which takes the duration of stockouts into consideration. They develop a model where the nonlinear functions of expected backlog and expected inventory on hand are approximated by use of piecewise linear segments. The resulting piecewise linear model is solved with a MIP-based heuristic.

In the current paper we extend the linearized MIP model of [7] through the introduction of \( \beta \) service level constraints, which are more common in industrial practice. In addition, we extend the basic model through the introduction of setup carry-overs, which lead to a better representation of the dynamic lot sizing problem. We apply a Fix&Optimize heuristic to solve a large number of test problem instances and compare the results with the results of the column generation heuristic proposed by Tempelmeier [14].

### 3. Problem statement

We consider \( K \) products that are produced to stock on a single resource with limited period capacities \( b_t \) (\( t = 1, 2, \ldots, T \)). The planning situation is completely identical with the classical CLSP with...
one exception: For each product $k$, the period demands $D_{kt}$ are random variables with given expected values $E(D_{kt})$ and variances $V(D_{kt})$ ($t = 1, 2, ..., T$). These data, which may vary over time, are the outcome of a forecasting procedure. It is assumed that the demands of the products are mutually independent and not autocorrelated. Unfilled demands are backordered and the amount of backorders is controlled by imposing a fill rate constraint, namely the $\beta_k$ service level.

At the beginning of the planning horizon there is a known initial inventory $Y_{k0}$ ($k = 1, 2, ..., K$) which may be zero. As mentioned above, we assume that the static-uncertainty strategy is in place. At the beginning of the planning horizon, the complete production plan is fixed, including the timing and the size of the cumulated production quantities. Inventory holding costs are charged on the inventory at the end of each period.

4. Model formulation

4.1. Basic model

In Table 1 we provide the symbols used in the following. For the evaluation of a production plan, the expected inventory at the end of any period $t$ and the expected backorders, which newly occur within a production cycle are required. As usual, the backorders associated with a production cycle are calculated as the difference between the expected backlog at the end of the production cycle and the expected backlog at the beginning of the production cycle. The calculation of these quantities is accomplished as follows. Let $Q^{(t)}_k$ denote the total amount available to fill the cumulated demands of periods 1 to $t$ (initial inventory in period 1 plus cumulated quantity produced up to period $t$). It is well-known that the expected backlog as well as the expected inventory on hand are nonlinear functions of $Q^{(t)}_k$, as shown in Fig. 1. We linearize this function as follows. Let $Y^{(t)}_k$ denote the cumulated demand from period 1 up to period $t$ and let $f_Y^{(t)}$ be the associated density function. Consequently the expected inventory on hand at the end of period $t$ for product $k$ is equal to

$$E(I_{kt}^{(t)}) = \int_{0}^{Q^{(t)}_k} (Q^{(t)}_k - y) \cdot f_Y^{(t)}(y) \cdot dy$$

$$= Q^{(t)}_k - E(Y^{(t)}_k) + C_1^{(t)}(Q^{(t)}_k)$$

whereby $C_1^{(t)}(Q^{(t)}_k)$ denotes the first-order loss function of the random variable $Y^{(t)}_k$ w.r.t. the quantity $Q^{(t)}_k$.

For any period $t$, Eq. (8) is a non-linear function, which can be approximated with an arbitrary precision with a sufficient number of linear segments, see Ref. [7]. This is shown in Fig. 1 for the case of normally distributed demand with mean $\mu = 100$ and standard deviation $\sigma = 30$.

The function of the expected inventory can be approximated as follows. Define $L$ line segments with interval limits $u^t_\ell$ that mark the cumulated production up to period $t$. Let $u^t_0$ be the lower limit of the relevant region. Accordingly, the slope of the inventory on hand function for line segment $\ell$ is

$$\Delta_{\ell t}^c = \left\{ \begin{array}{l} \left[ u^t_\ell - E(Y^{(t)}_k) + G_1^{(t)}(u^t_\ell) \right] \left[ u^{t-1}_\ell - E(Y^{(t-1)}_k) + G_1^{(t-1)}(u^{t-1}_\ell) \right]^{-1} \\ \left[ u^t_\ell + G_1^{(t)}(u^t_\ell) \right] - \left[ u^{t-1}_\ell + G_1^{(t-1)}(u^{t-1}_\ell) \right] \\ \ell = 1, 2, ..., L; \\ k = 1, 2, ..., K; \\ t = 1, 2, ..., T \end{array} \right. \tag{9}$$

A backlog at the end of period $t$ for product $k$ occurs, if the cumulated demand up to period $t$, $Y^{(t)}_k$, is greater than the cumulated production quantity up to period $t$, $Q^{(t)}_k$. Hence, the expected backlog at the end of period $t$ for product $k$ is

$$E\left\{ I_{kt}^{(t)} \right\} = C_1^{(t)}(Q^{(t)}_k)$$

As usual, it is assumed that after production in period $t$, first the actual backlog is delivered. Particularly if the production quantity is small, it may be not sufficient to clear the complete backlog and hence some backlog quantity may remain. This backlog after production, but before fulfillment of the demand of period $t$ is the

![Fig. 1. Approximation of expected backlogs and inventory on hand.](image-url)
difference between the cumulated amount available $Q_{t}^{1}$ up to period $t$ and the cumulated demand up to period $t-1$. Its expected value is

$$E\left[ B_{ki}^{1}\right] = Q_{t}^{1}(t), \quad k = 1, 2, \ldots, K; t = 1, 2, \ldots, T$$  

(11)

The expected number of backorders $E\left[ B_{kt}^{1}\right]$ for product $k$ in period $t$ can then be expressed as the difference between the backlog at the end of period $t$ and the backlog immediately after production in period $t$.

$$E\left[ B_{kt}^{1}\right] = E\left[ B_{kt}^{1}\right] - E\left[ B_{kt}^{1}\right] \quad k = 1, 2, \ldots, K; t = 1, 2, \ldots, T$$  

(12)

Similar to Eq. (9), the non-linear function of the backorders at the end of period $t$ can be approximated, whereby the slopes are

$$\Delta^{b}_{kt} = \left[ G_{t}(u_{kt}^{*}) - G_{t}(u_{kt}^{*}) \right] - \left[ G_{t}(u_{kt}^{*}) - G_{t}(u_{kt}^{*}) \right]$$  

(13)

The $\Delta^{b}_{kt}$ values as well as the $\Delta^{b}_{kt}$ values are data used in the MIP-models which will be introduced in the following. Let $w_{kt}^{*}$ be the production quantity of product $k$ in period $t$ associated with interval $\tau$. As the inventory function is convex, $w_{kt}^{*}$ is only positive if $w_{kt}^{*} = \sum_{\tau = 1}^{t} w_{kt}^{*}$. Hence, $Q_{t}^{1} = \sum_{\tau = 1}^{t} w_{kt}^{*}$ is the cumulated production quantity of product $k$ up to period $t$. Let $\Delta^{b}_{kt}$ be the expected inventory and $\Delta^{b}_{kt}$ the expected backorder at point $u_{kt}^{*}$. Then the following linear MIP model is obtained:

Minimize $E(C) = \sum_{k = 1}^{K} \sum_{t = 1}^{T} \left( s_{kt} \cdot \gamma_{kt} + h_{kt} \cdot \Delta^{b}_{kt} + \sum_{\tau = 1}^{t} \Delta^{b}_{kt} \cdot w_{kt}^{*} \right)$  

(14)

subject to

$$\sum_{k = 1}^{K} \left( t_{kt} \cdot q_{kt} + t_{kt} \cdot \gamma_{kt} \right) \leq b_{kt} \quad t = 1, 2, \ldots, T$$  

(15)

$$\sum_{\tau = 1}^{t} w_{kt} \leq w_{kt}^{*} \leq w_{kt}^{*} \quad t = 2, 3, \ldots, T; k = 1, 2, \ldots, K$$  

(16)

$$\gamma_{kt} \leq \gamma_{kt} \quad t = 1, 2, \ldots, T; k = 1, 2, \ldots, K$$  

(17)

$$\gamma_{kt} \leq \gamma_{kt} \quad t = 1, 2, \ldots, T; k = 1, 2, \ldots, K$$  

(18)

$$q_{kt} \leq M \cdot \gamma_{kt} \quad t = 1, 2, \ldots, T; k = 1, 2, \ldots, K$$  

(19)

(Service level constraints, to be specified)  

(20)

$$\gamma_{kt} \in [0, 1] \quad t = 1, 2, \ldots, T; k = 1, 2, \ldots, K$$  

(21)

$$W_{kt}^{*} \geq 0 \quad t = 1, 2, \ldots, T; \tau = 1, 2, \ldots, L; k = 1, 2, \ldots, K$$  

(22)

The precise specification of the service level constraints depends on the type of service level used.

$\beta_t$, Service Level: If the finite horizon $\beta_t$ criterion is used, then constraint (20) is specified as

$$1 - \frac{\sum_{t = 1}^{T} \left( \Delta^{b}_{kt} + \sum_{\tau = 1}^{t} \Delta^{b}_{kt} \cdot w_{kt}^{*} \right)}{\sum_{t = 1}^{T} E[B_{kt}]} \geq \beta^{*}_{t} \quad k = 1, 2, \ldots, K$$  

(23)

$\beta_t$, Service Level: For the case of the cycle $\beta_c$ service level, constraint (20) is replaced with

$$1 - \frac{\sum_{t = 1}^{T} \left( \Delta^{b}_{kt} + \sum_{\tau = 1}^{t} \Delta^{b}_{kt} \cdot w_{kt}^{*} \right)}{\sum_{t = 1}^{T} E[B_{kt}]} \geq \beta^{*}_{t} \quad k = 1, 2, \ldots, K$$  

(24)

$$1 - \frac{\sum_{t = 1}^{T} \left( \Delta^{b}_{kt} + \sum_{\tau = 1}^{t} \Delta^{b}_{kt} \cdot w_{kt}^{*} \right)}{\sum_{t = 1}^{T} E[B_{kt}]} \leq \beta^{*}_{t} \quad k = 1, 2, \ldots, K$$  

(25)

$$\gamma_{kt+1} = 1 \quad k = 1, 2, \ldots, K$$  

(26)

According to the relationship between $\beta_c$ and $\beta_t$ discussed above, constraints (24) and (25) ensure the target cycle $\beta_c$ service level is met only if the production quantities are set such that the $\beta_t$ service levels observed in the final periods of all production cycles are equal. However, considering scarce capacities, the resulting production plan will not have this characteristic. In case of limited capacity, production is shifted into an earlier production cycle. As a consequence, the inventory level in this particular cycle is greater than required. Consequently, the target $\beta_c$ service level is exceeded. In this situation, the $\beta_t$ service level cannot be exactly met by imposing the $\beta_t$ constraints. However, constraints (25) and (26) guarantee that $\beta_t$ is not underachieved.

4.2. Model with setup carry-overs

The above basic model formulation can easily be extended to incorporate setup carry-overs. In this case, the model recognizes that a setup may be omitted if a product-specific setup state of the resource is carried over to the next period. This extension has first been introduced for deterministic lot sizing models (see [5,12]). Setup carry-overs are introduced with the help of the variables shown in Table 2.

We extend the above model by the following constraints:

$$\sum_{k = 1}^{K} w_{kt} \leq \sum_{k = 1}^{K} w_{kt} \quad t = 1, 2, \ldots, T$$  

(27)

$$\sum_{k = 1}^{K} \omega_{kt} \leq 1 \quad t = 1, 2, \ldots, T$$  

(28)

$$\omega_{kt} \leq \gamma_{kt+1} + \omega_{kt+1} \quad k = 1, 2, \ldots, K; t = 2, 3, \ldots, T$$  

(29)

$$\omega_{kt+1} \leq \omega_{kt} + \gamma_{kt} + \omega_{kt} \quad k = 1, 2, \ldots, K; t = 1, 2, \ldots, T$$  

(30)

$$\gamma_{kt} = 0 \quad t = 1, 2, \ldots, T$$  

(31)

$$\omega_{kt} \leq 0 \quad t = 1, 2, \ldots, T$$  

(32)

$$\gamma_{kt} \leq \omega_{kt} \quad k = 1, 2, \ldots, K$$  

(33)

The binary variables $\omega_{kt}$ indicate that product $k$ was produced at the end of period $(t - 1)$ or in an earlier period and the setup state of the resource is carried over from that period to period $t$. The continuous variable $v_t$ ensures that there must be no production of any other product in period $t$, if the resource is already setup for a given product $k$ at the beginning of the two consecutive periods $t$ and $t+1$ (this means no setup occurs in period $t$). In particular, (30) forces $v_t$ to zero, if there is a setup in period $t$. As a consequence, the setup state variables $\omega_{kt+1}$ on the left of (29) becomes zero. It is well known that the inclusion of setup

<table>
<thead>
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<th>Table 2</th>
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<tr>
<td>Additional notation to include setup carry-overs.</td>
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</tbody>
</table>

- $\omega_{kt} \in [0, 1]$ binary variable which indicates that the resource is in the setup state for product $k$ at the beginning of period $t$
- $v_t$ indicator variable which takes the value 1 if there is no setup in period $t$
carry-overs into the CLSP usually leads to better and more realistic production plans.

5. Solution approach

Even for small problem instances the exact solution of the above basic model and even more of the extended model is usually too time-consuming. For problem sizes of practical dimensions, we propose to apply a variant of the Fix&Optimize heuristic (see [6]), which is also known as “exchange heuristic” (see [9]). Basically, it is a MIP-based heuristic which decomposes the overall lot sizing model with many binary setup (and setup state) variables into smaller subproblems, each with significantly less binary variables. The subproblems are generated according to the specific characteristics of the problem type under consideration. The main procedure iteratively generates a sequence of subproblems which are solved by a MIP solver. Thereby, a new subproblem is constructed under consideration of the solutions found in the preceding iterations. It is a particular characteristic of the Fix&Optimize heuristic, that each solution is feasible with respect to the binary variables. Therefore in this respect, it represents a feasible production plan. As some subproblem definitions may not have a feasible solution w.r.t. the capacity constraints, we include (dummy) overtime variables with associated prohibitively high costs. Hence, the MIP-solver always finds a feasible solution for a subproblem, which, however, in case of positive overtime is excluded from further consideration due to its high objective value.

We propose the Fix&Optimize heuristic as it has shown to deliver high-quality results, is easy to implement and can be adjusted to fit with different kinds of mixed-integer lot sizing problems (see [6]).

The general structure of the heuristic for the solution of the basic model SCLSP is shown in Fig. 2 with the notation given in Table 3.

The set of subproblems is generated according to a product-oriented decomposition. Each subproblem corresponds to a given number of products considered over the complete planning horizon $T$. For each subproblem $s (s = 1, ..., S)$ the binary variables $γ_{kt}$ are divided into the sets $KT^s_{fix}$ (with fixed $γ_{kt}$-values) and $KT^s_{opt}$ (with $γ_{kt}$-values to be optimized by the MIP solver). The binary variables in the set $KT^s_{fix}$ are set to a given setup pattern $P^s_{fix}$ (procedure Fix()). If the optimum objective value of the current subproblem, $Z_s$, is less than or equal to the objective value of the currently best solution, $Z^*$, then the corresponding solution is saved and used to define the set $KT^s_{fix}$ for the next subproblem.

6. Numerical results

In order to test the quality of the proposed heuristic, we conducted a numerical experiment which is based on problem instances presented by Tempelmeier [14].

We generated problem instances with 10 and 20 periods as well as 10 and 40 products. For each combination of the parameters $T, K$, TBO, $β_i$ (the same fill rate is used for all products) and capacity, ten replications were randomly generated, whereby the expected demands per period and period were drawn from a continuous uniform distribution. For each product, the period demands are assumed to be normally distributed with the parameters shown in Table 4. Note that in contrast to the description in [14], the TBOs were not drawn randomly but are given as deterministically set.

In total, we solved 1080 problem instances. The objective values found with the proposed MIP model are compared to the results obtained with the ABC$γ_i$ heuristic presented by Tempelmeier and Herpers [16] and with the column generation heuristic proposed by Tempelmeier [14]. Furthermore, setup times are neglected, as there are no benchmark solutions available for this problem characteristic. As these papers provide benchmark results only for problem instances with a $β_i$ service level constraint, we confine our numerical tests to this criterion. However, we performed a limited number of exploratory computations and found that the duality gaps for the model variant using $β_i$ are almost the same as for the model variant using $β_c$.

The holding costs $h$ of product $k$ are calculated as $h_k = 2 \cdot s / \overline{d_k} \cdot TBO^2_k$, where $\overline{d_k}$ is the mean demand per period for this product. The setup costs were $s = 500$. The capacity is calculated as product of the average workload $w_k = \sum_{e=1}^{E-k} d_e$ and the workload factor. The resulting capacity $b_i$ is $b_i^{low} = 1.1 \cdot w$, $b_i^{medium} = 1.5 \cdot w$ and $b_i^{high} = 2 \cdot w$. As the results as well as the solution times depend on the number of line segments used in the MIP model, we solved each of the 1080 problem instances with 10, 18 and 34 line segments. The specific bounds of the line segments are calculated with a convex curve approximation method known as block sandwich algorithm (see [13]). The numerical experiment showed that 18 line segments provided a good compromise between accuracy and solution time. Table 5 shows the relative deviations of the costs obtained with the Fix&Optimize (F&O) heuristic compared to the solutions calculated with the column generation heuristic (CG) and the ABC$γ_i$ heuristic. Positive values indicate that the Fix&Optimize heuristic performs worse than its competitor. It turns out that the new heuristic performs significantly better than the ABC$γ_i$ heuristic in all cases. In contrast, the comparison with the column generation heuristic shows that the new heuristic performs only better for the small problems with 10 products and low capacities. This observation is in line with the behaviour of the
overs exist in many real-life lot sizing problems, and that lot sizing medium and high. Considering that setup times as well as setup carry-overs, at least when capacities are of costs decreases with the problem size. Within 600 s, CPLEX characteristic of\[14\] appears to be competitive for problem instances without setup times and without setup carry-overs, at least when capacities are significantly lower.

In Table 6 the average solution times of the Fix&Optimize heuristic compared to the CPLEX solver after 600 s.

Table 4 Parameters for the numerical experiment.

| Number of periods, T | 10, 20 |
| Number of products, K | 10, 40 |
| Capacity | Low, medium, high |
| Time-between-orders, TBO | 1, 5, 10 |
| Target fill rate \( \beta \) | 0.80, 0.90, 0.98 |
| Mean period demand | Continuous uniform \( U(0, 100) \) |
| Coefficient of variation | \( U(0.15, 0.20, 0.25, 0.30, 0.35) \) |

Table 5 Objective values found with the Fix&Optimize heuristic compared to the column generation and the ABC\(_2\) heuristic.

<table>
<thead>
<tr>
<th>Capacity (w)</th>
<th>( T_{\text{fl}} )</th>
<th>( T_{\text{opt}} )</th>
<th>( T_{\text{CG}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K = 10 )</td>
<td>1.10 + 13.21 + 23.56 + 18.39 + 19.24 + 27.36 + 23.15</td>
<td>1.50 + 2.90 + 5.75 + 4.33 + 11.19 + 10.40 + 10.80</td>
<td>2.00 + 0.95 + 1.34 + 0.15 + 8.26 + 6.93 + 7.60</td>
</tr>
<tr>
<td>( K = 40 )</td>
<td>1.10 + 0.84 + 0.94 + 0.89 + 26.74 + 24.20 + 25.47</td>
<td>1.50 + 1.63 + 2.60 + 2.12 + 16.52 + 12.02 + 14.27</td>
<td>2.00 + 2.04 + 3.36 + 2.70 + 11.92 + 7.43 + 9.70</td>
</tr>
</tbody>
</table>

Table 3 Notation.

| \( \gamma_k \) | Value of the fixed setup variable \( \gamma_k \) |
| \( \gamma_{sk} \) | Optimal value of the setup variable \( \gamma_{sk} \) in subproblem \( s \) |
| \( L \) | Number of replications |
| \( S \) | Number of subproblems |
| \( KT \) | Set of all combinations of product and period indices |
| \( KT_{\text{fix}} \) | Set of indices of products and periods for which the setup variables \( \gamma_{sk} \) are optimized in subproblem \( s \) |
| \( KT_{\text{opt}} \) | Set of indices of products and periods for which the setup variables \( \gamma_{sk} \) are fixed in subproblem \( s \) |

Table 6 Comparison of solution times (seconds) for Fix&Optimize heuristic and column generation heuristic.

<table>
<thead>
<tr>
<th>Capacity (w)</th>
<th>F&amp;O</th>
<th>CG</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 10 )</td>
<td>( T = 20 )</td>
<td>( T = 10 )</td>
</tr>
<tr>
<td>( K = 10 )</td>
<td>20.19</td>
<td>126.92</td>
</tr>
<tr>
<td>( K = 40 )</td>
<td>19.63</td>
<td>119.29</td>
</tr>
</tbody>
</table>

Table 7 Relative cost reduction and solution times of the Fix&Optimize heuristic compared to the CPLEX solver after 600 s.

<table>
<thead>
<tr>
<th>Capacity (w)</th>
<th>Costs</th>
<th>Solution times</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 10 )</td>
<td>( T = 20 )</td>
<td>( T = 10 )</td>
</tr>
<tr>
<td>( K = 10 )</td>
<td>1.11 + 8.14 + 93.62 + 44.38</td>
<td></td>
</tr>
<tr>
<td>( K = 40 )</td>
<td>0.44 + 5.91 + 88.04 + 67.12</td>
<td></td>
</tr>
</tbody>
</table>

In Table 6 the average solution times of the Fix&Optimize heuristic are compared to the solution times of the column generation heuristic.

The results show, that the new Fix&Optimize heuristic provides better solutions than the column generation heuristic in cases with high capacity utilization and a small number of products. For a large number of products, the Fix&Optimize heuristic generates (slightly) better results only in cases of high utilizations. In the other cases, the column generation heuristic is better. In all cases, the computation times of the column generation heuristic are significantly lower.

For the extended model with setup carry-overs, there exists currently no alternative solution procedure. Consequently, a set of benchmark problems is not available yet. Therefore, we compared the solutions found with the Fix&Optimize heuristic to the solution obtained with the CPLEX solver after 600 s. Table 7 shows, that the relative performance of CPLEX in terms of costs decreases with the problem size. Within 600 s, CPLEX found better solutions for the problem instances with planning horizon \( T = 10 \). For \( T = 20 \), in most cases the heuristic performed better. It is not surprising, that the heuristic required less computation time than CPLEX.

To summarize, we are surprised that the column generation heuristic of\[14\] appears to be competitive for problem instances without setup times and without setup carry-overs, at least when capacities are medium and high. Considering that setup times as well as setup carry-overs exist in many real-life lot sizing problems, and that lot sizing issues mainly arise when capacity is scarce, the modeling and solution approach proposed in this paper provides considerably more flexibility.

In addition, it is a great virtue of MIP-based heuristics, that the underlying optimization models can be easily modified to meet additional practical requirements without significant modifications of the basic structure of the solution procedure.

### 7. Conclusion

In this paper we introduced an MIP-based approach to model dynamic capacitated lot sizing problems with stochastic demand under a fill rate constraint. We solved the model with a Fix & Optimize heuristic. For a given set of experimental data with up to 20 periods and 40 products we showed that the heuristic provides better solutions than the column generation heuristic in cases with a small number of products and in cases with high capacity utilizations.

For larger problem sizes, the column generation heuristic performed better. Nevertheless, it must be kept in mind, that most real-life lot sizing problems include setup times, which are handled by the Fix&Optimize heuristic, but which cannot be captured by the column generation heuristic.

The basic model has been extended to account for setup carry-overs and the Fix & Optimize heuristic can be used to solve this kind of model, too.
Other extensions, such as multiple parallel resources, are also possible. These are the subject of future research efforts.

References